Solution to Assignment 7

Section 7.3

10. We let $F(t) = \int_a^t f$. Then $G(x) = \int_a^{\nu(x)} f = F(\nu(x))$. Applying the Chain Rule and then the Second Fundamental Theorem,

$$G'(x) = F'(\nu(x))\nu'(x) = f(\nu(x))\nu'(x) \ .$$

11. First,

$$F(x) = \int_0^{x^2} \frac{1}{1+t^3} dt \ .$$

By taking $\nu(x) = x^2$ and applying the previous problem, we have

$$F'(x) = \frac{1}{1+x^6} \times 2x = \frac{2x}{1+x^6} \ .$$

Next, write

$$F(x) = \int_0^x \sqrt{1+t^2} dt - \int_0^{x^2} \sqrt{1+t^2} dt \ ,$$

and apply the previous problem separately to get

$$F'(x) = \sqrt{1+x^2} - 2x\sqrt{1+x^4} \ .$$

16. Differentiate both sides to get

$$f(x) = -f(x) \ ,$$

after noting

$$\int_x^1 f = \int_0^1 f - \int_0^x f \ .$$

Check the assumption for the Second Fundamental Theorem.

Supplementary Exercise

1. Evaluate the following integrals

$$\int_0^a x^2 \sqrt{a^2 - x^2} dx \ .$$

Solution. WLOG take $a > 0$. Use the change of variables $x = a\sin\theta,\ \theta \in [0, \pi/2]$. Then $dx/d\theta = a\cos\theta$ on $[0, \pi/2]$.

$$\begin{aligned}
\int_0^a x^2\sqrt{a^2 - x^2}\ dx &= \int_0^{\pi/2} a^2\sin^2\theta(|\ a\ |\cos\theta)(a\cos\theta)\ d\theta \\
&= a^4\int_0^{\pi/2}\sin^2\theta\cos^2\theta\ d\theta \\
&= \frac{a^4}{4}\int_0^{\pi/2}\sin^2 2\theta\ d\theta \\
&= \frac{a^4}{4}\int_0^{\pi/2}\left(\frac{1-\cos 4\theta}{2}\right)\ d\theta \\
&= \frac{a^4}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{\pi/2} \\
&= \frac{\pi a^4}{16}\ .
\end{aligned}$$

2. Prove the following formula: For any "nice" function f

$$\int_0^\pi xf(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx.$$

Solution.

$$\begin{aligned}
\int_0^\pi xf(\sin x)\ dx &= \int_0^{\pi/2} xf(\sin x)\ dx + \int_{\pi/2}^\pi xf(\sin x)\ dx \\
&= \int_0^{\pi/2} xf(\sin x)\ dx + \int_{\pi/2}^0 (\pi - u)f(\sin(\pi - u))(-1)\ du \\
&= \int_0^{\pi/2} xf(\sin x)\ dx + \int_0^{\pi/2}(\pi - x)f(\sin x)\ dx = \pi\int_0^{\pi/2} f(\sin x)\ dx\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^\pi f(\sin x)\ dx &= \int_0^{\pi/2} f(\sin x)\ dx + \int_{\pi/2}^0 f(\sin(\pi - u))(-1)\ du \\
&= 2\int_0^{\pi/2} f(\sin x)\ dx\ .
\end{aligned}$$

Hence,

$$\int_0^\pi xf(\sin x)\ dx = \frac{\pi}{2}\int_0^\pi f(\sin x)\ dx\ .$$

3. Evaluate the integral

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}dx.$$

Hint: Use the previous problem.

Solution.

$$\begin{aligned}
\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx &= \int_0^\pi \frac{x\sin x}{2-\sin^2 x}\,dx \\
&= \frac{\pi}{2}\int_0^\pi \frac{\sin x}{2-\sin^2 x}\,dx \\
&= -\frac{\pi}{2}\int_0^\pi \frac{-\sin x}{1+\cos^2 x}\,dx \\
&= \frac{-\frac{\pi}{2}\int_0^\pi d(\cos x)}{1+\cos^2 x} \\
&= -\frac{\pi}{2}\operatorname{Arctan}\cos x\Big|_0^\pi \\
&= \frac{\pi^2}{4}\ .
\end{aligned}$$

4. For a continuous function f on $[-a,a]$, prove that when it satisfies

$$\int_{-a}^a fg = 0,$$

for all even, integrable functions g, it must be an odd function.

Solution. Step 1. Define:

$$\begin{aligned}
f &= f_e + f_o \\
f_e &= \frac{f(x)+f(-x)}{2} \\
f_o &= \frac{f(x)-f(-x)}{2}
\end{aligned}$$

Note f_e is even while f_o is odd.
Then,

$$0 = \int_{-a}^a fg = \int_{-a}^a f_e g + \int_{-a}^a f_o g\ .$$

Use change of variables,

$$\begin{aligned}
\int_{-a}^0 f_o g &= \int_{-a}^0 f_o(x)g(x)dx \\
&= \int_{-a}^0 f_o(-x)g(-x)dx \\
&= \int_0^a f_o(-x)g(x)dx \\
&= -\int_0^a f_o(x)g(x)dx\ .
\end{aligned}$$

Therefore, $\int_{-a}^a f_o g = 0$.
It follows that

$$0 = \int_{-a}^a f_e g\ .$$

As f_e is even, set $g = f_e$, $\int_{-a}^a {f_e}^2 = 0 \Rightarrow f_e \equiv 0$, so $f = f_o$ is odd. At the last we use the continuity of f (so are f_e and f_o).

5. Evaluate the following integrals:

(a)

$$\int_0^\pi x \sin x dx \ ,$$

(b)

$$\int_0^1 \text{Arccos} x dx.$$

The inverse cosine function Arccos maps $[-1,1]$ to $[0,\pi]$.

Solution. (a)

$$\int_0^\pi x \sin x \ dx = -\int_0^\pi x \ d(\cos x) = (-x\cos x)\Big|_0^\pi + \int_0^\pi \cos x \ dx = \pi \ .$$

(b) Let $x = \cos\theta,\ \theta \in [0,\pi/2]$. Then $dx/d\theta = -\sin\theta$ on $[0,\pi/2]$. Then

$$\begin{aligned}\int_0^1 \text{Arccos } x \ dx &= -\int_0^{\pi/2} \theta \ d(\cos\theta) \\ &= (-\theta\cos\theta)\Big|_0^{\pi/2} + \int_0^{\pi/2} \cos\theta \ d\theta \\ &= 1 \ .\end{aligned}$$

6. Evaluate the following integrals:

(a)

$$\int_0^1 (1-x^2)^n dx \ ,$$

(b)

$$\int_0^1 x^m (\log x)^n dx, \quad m,n \in \mathbb{N}.$$

Solution. (a)

Let $x = \sin\theta,\ \theta \in [0,\pi/2]$. Then $dx/d\theta = \cos\theta$ on $[0,\pi/2]$.

$$\begin{aligned}I_n \equiv \int_0^1 (1-x^2)^n \ dx &= \int_0^{\pi/2} \cos^{2n+1}\theta \ d\theta \\ &= \int_0^{\pi/2} \cos^{2n}\theta \ d(\sin\theta) \\ &= (\cos^{2n}\theta\sin\theta)\Big|_0^{\pi/2} + 2n\int_0^{\pi/2} \cos^{2n-1}\theta\sin^2\theta \ d\theta \\ &= 2n\int_0^{\pi/2} \cos^{2n-1}\theta(1-\cos^2\theta) \ d\theta \\ &= 2n\int_0^{\pi/2} \cos^{2n-1}\theta \ d\theta - 2n\int_0^{\pi/2} \cos^{2n+1}\theta \ d\theta = 2nI_{n-1} - 2nI_n \ .\end{aligned}$$

Therefore,

$$\begin{aligned} I_n &= \frac{2n}{2n+1} I_{n-1} \\ &= \frac{2n}{2n+1}\frac{2n-2}{2n-1}\cdots\frac{2}{3}\, I_0 \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!}\int_0^{\pi/2}\cos\theta\; d\theta \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!}\sin\theta\Big|_0^{\pi/2} \\ &= \frac{2^{2n}(n!)^2}{(2n+1)!}\ . \end{aligned}$$

(b)

$$\begin{aligned} I_{m,n} \equiv \int_0^1 x^m(\log x)^n\; dx &= \frac{1}{m+1}\lim_{\varepsilon\to 0}\int_\varepsilon^1 (\log x)^n d(x^{m+1}) \\ &= \lim_{\varepsilon\to 0}\frac{x^{m+1}}{m+1}(\log x)^n\Big|_\varepsilon^1 - \frac{1}{m+1}\int_0^1 x^{m+1}(n(\log x)^{n-1})\frac{1}{x}\; dx \\ &= -\frac{n}{m+1}\int_0^1 x^m(\ln x)^{n-1}\; dx \\ &= -\frac{n}{m+1} I_{m,n-1} \\ &= (-1)^n\frac{n!}{(m+1)^n} I_{m,0} \\ &= (-1)^n\frac{n!}{(m+1)^n}\int_0^1 x^m\; dx \\ &= \frac{(-1)^n n!}{(m+1)^n}\frac{x^{m+1}}{m+1}\Big|_0^1 \\ &= \frac{(-1)^n n!}{(m+1)^{n+1}}\ . \end{aligned}$$

7. Study the uniform convergence for the following sequences of functions. Find the pointwise limits first.

   (a) $\left\{\dfrac{x}{x+n}\right\}$; $\quad [0,\infty), \quad [0,12]$.

   (b) $\left\{\dfrac{x^n}{1+x^n}\right\}$; $\quad [0,\infty), \quad [0,1], \quad [2,5]$.

Solution. (a) Let f_n be the sequence. We have $f_n'(x) = n/(x+n)^2 > 0$ which means the function is increasing. So $\|f_n - 0\|_\infty = \|f_n\|_\infty = 1$, which is not equal to zero. This sequence is not uniformly convergent to 0 on $[0,\infty)$. If now we restrict to $[0,12]$, the max of f_n is attained at $x=12$, so now $\|f_n\|_\infty = 12/(12+n) \to 0$ as $n\to\infty$. We conclude that it is uniformly convergent on $[0,12]$.

(b) The pointwise limit is the constant one for $x \in (1,\infty)$. We have

$$\frac{d}{dx}\left(1-\frac{x^n}{1+x^n}\right) = \frac{d}{dx}\frac{1}{1+x^n} = \frac{-nx^{n-1}}{(1+x^n)^2} < 0, \quad x\in(0,\infty)\ ,$$

so the supnorm is given by $\lim_{x\to 1} 1/(1+x^n) = 1/2$. $\|f_n - 1\|_\infty = 1/2$. It means $\|f_n - 1\|_\infty = 1/2 \neq 0$, so no uniform convergence on $(1,\infty)$. On the other hand, on $[2,5]$ the supnorm is attained at $x = 2$, so $\|f_n - 1\|_\infty = 1/(1+2^n) \rightrightarrows 0$.

8. Study the uniform convergence of the following sequence of functions by any method.

   (a) $\left\{ \dfrac{nx}{1+n^2x^2} \right\}; \quad [0,\infty)$.

   (b) $\left\{ \dfrac{\sin nx}{1+nx} \right\}; \quad [0,\infty), \quad [1,\infty)$.

   Solution. (a) The pointwise limit is the zero function. By taking derivative we see that the maximum of f_n is attained at $x = 1/n$. It follows that

   $$\left\| \frac{nx}{1+n^2n^2} - 0 \right\| = \frac{n \times 1/n}{1+n^2 \times 1/n^2} = \frac{1}{2} \neq 0 \ ,$$

   so the convergence is not uniform.

   (b) The pointwise limit is again the zero function. It is not good to determine the maximum of each function. But we observe that $f_n(\pi/(2n)) = 2/(2+\pi)$, so

   $$\left\| \frac{\sin nx}{1+nx} - 0 \right\| \geq f_n\left(\frac{\pi}{2n}\right) = \frac{2}{2+\pi} \neq 0 \ ,$$

   so the convergence is not uniform. In this case it is nice to draw an ε-tube with $\varepsilon = 1/4$, say, to visualize the situation.

9. Study the pointwise and uniform convergence of $\left\{ n^\alpha x^\beta e^{-nx} \right\}$ on $[0,\infty)$ for $\alpha, \beta > 0$.

   Solution. (c) The pointwise limit is the zero function on $[0,\infty)$. We find the maximum of $f_n = n^\alpha x^\beta e^{-nx}$ by setting

   $$0 = \frac{d}{dx} f_n(x) = n^\alpha \beta x^{\beta-1} e^{-nx} - n^{\alpha+1} x^\beta e^{-nx} = 0,$$

   which implies $x = \beta/n$. It is easy to check that this is the maximum as f_n is positive and tends to 0 at $x = 0$ and $x = \infty$. Therefore,

   $$\|x^2 e^{-nx} - 0\| = f_n(\beta/n) = \beta^\beta n^{\alpha-\beta} e^{-\beta} \ ,$$

   which tends to 0 if and only if $\alpha < \beta$. We conclude that $\{n^\alpha x^\beta e^{-nx}\}$ uniformly converges to 0 on $[0,\infty)$ iff $\alpha < \beta$.